

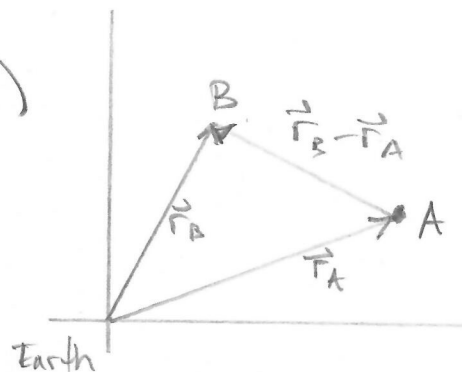
The Cosmological Principle

C1

The Universe is isotropic (looks the same in every direction) and homogeneous (looks the same from every location)

The expansion of the Universe looks the same from every location.

$$\begin{aligned} \vec{V}_A &= H_0 \vec{r}_A \\ \vec{V}_B &= H_0 \vec{r}_B \end{aligned} \quad \left| \rightarrow \quad \vec{V}_B - \vec{V}_A = H_0 (\vec{r}_B - \vec{r}_A) \right.$$



From point A the Universe is expanding in the same way as from Earth. (ie the relative velocity of B with respect to A is proportional to the distance between A and B)

Pressureless Dust Model

Model: The Universe is filled with a pressureless "dust" of uniform density $\rho(t)$. Choose an arbitrary point for the origin. In this model "dust" represents all the matter in the Universe. As the Universe expands the dust is carried radially outward from the origin.

Consider a shell containing a mass m of dust that is expanding outward with a radial velocity of $v(t) = \frac{dr(t)}{dt}$

This shell always contains the same amount of gas.

The total mechanical energy of the shell is:

$$K(t) + U(t) = E(t) \quad (1)$$

The total energy is written as:

$$E = -\frac{1}{2} m k c^2 \bar{w}^2$$

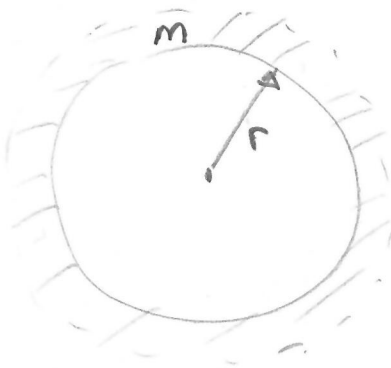
where $\bar{w}$ is the radius of the shell at the current time

$$\bar{w} = r(t_0)$$

$$(1) \rightarrow \frac{1}{2} m v^2(t) - \frac{G M_r m}{r(t)} = -\frac{1}{2} m k c^2 \bar{w}^2 \quad (2)$$

$$\text{where } M_r = \frac{4}{3} \pi r^3(t) \rho(t)$$

M_r is the total mass within a radius r
 m is the mass in the shell



$$(2) \rightarrow v^2(t) - \frac{2 \cdot G}{r(t)} \cdot \frac{4}{3} \pi r^3(t) \rho(t) = -k c^2 \bar{w}^2$$

$$\rightarrow v^2(t) - \frac{8}{3} \pi G \rho(t) r^2(t) = -k c^2 \bar{w}^2 \quad (3)$$

$k=0$: The total energy $E=0 \rightarrow$ expansion slows down and halts at $t \rightarrow \infty$ C3

$k>0$: The total energy is negative $\rightarrow$ The Universe is bound

$k<0$: The total energy is positive $\rightarrow$ The Universe is unbound.

Coordinate Distance

$$r(t) = R(t) r(t_0) = R(t) \bar{w} \quad (4)$$

Where $r(t_0)$ is the distance between us and an object now
 $r(t)$ is the distance between us and the object in the past

$R(t)$ is the scale factor

$$\text{for } t=t_0 : r(t_0) = R(t_0) r(t_0) \rightarrow R(t_0) = 1$$

The scale factor now = 1

Recall definition of redshift: $z = \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}} = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} - 1$

$$\Rightarrow \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = 1+z \Rightarrow \frac{\lambda_{\text{emit}}}{\lambda_{\text{obs}}} = \frac{1}{1+z}$$

$$(4) \rightarrow R(t) = \frac{r(t)}{r(t_0)} = \frac{\lambda_{\text{emit}}}{\lambda_{\text{obs}}} = \frac{1}{1+z} \quad (5)$$

M_r is constant with time (ie. The total mass with the radius r is constant)

C4

$$M_r \propto \rho(t) r^3(t) = \rho(t) R^3(t) \underbrace{r^3(t_0)}_{\text{constant}}$$

$$\rho(t) R^3(t) = \rho(t_0) R^3(t_0) = \rho(t_0)$$

$$\rho(t) = \frac{\rho(t_0)}{R^3(t)} \xrightarrow{\textcircled{5}} \rho(z) = \rho_0 (1+z)^3 \quad \textcircled{6}$$

$\uparrow$ larger density at redshift z $\uparrow$ density now

Evolution of Pressureless Dust Universe

$$V(t) = H(t) r(t) = H(t) \cdot R(t) \cdot \underbrace{r(t_0)}_{\substack{\text{scale} \\ \text{factor}}} \quad \leftarrow \text{radius now}$$

$$V(t) = \frac{dr(t)}{dt} = r(t_0) \frac{dR(t)}{dt}$$

$$H(t) = \frac{V(t)}{R(t) \cdot r(t_0)} = \frac{r(t_0) \frac{dR(t)}{dt}}{R(t) \cdot r(t_0)} = \frac{1}{R(t)} \frac{dR(t)}{dt} \quad \textcircled{7}$$

From the $K + U = E$ equation (3) $\Rightarrow$

C5

$$v^2(t) - \frac{8}{3} \pi G \rho(t) r^2(t) = -kc^2 r^2(t) \Rightarrow$$

$$[H(t) R(t) \cancel{r(t_0)}]^2 - \frac{8}{3} \pi G \rho(t) R^2(t) \cancel{r^2(t_0)} = -kc^2 \cancel{r^2(t_0)}$$

$$R^2(t) \left[H^2(t) - \frac{8}{3} \pi G \rho(t) \right] = -kc^2 \quad (8) \quad \Rightarrow$$

$$\left[\left(\frac{1}{R(t)} \frac{dR(t)}{dt} \right)^2 - \frac{8}{3} \pi G \rho(t) \right] R^2(t) = -kc^2$$

$$M_r = \text{constant} \rightarrow \rho(t) \cdot r^3(t) = \rho(t_0) r^3(t_0) \Rightarrow$$

$$\rho(t) R^3(t) r^3(t_0) = \rho(t_0) r^3(t_0) \Rightarrow$$

$$\rho(t) = \frac{\rho(t_0)}{R^3(t)}$$

$$\boxed{\left(\frac{dR(t)}{dt} \right)^2 - \frac{8}{3} \pi G \frac{\rho_0}{R(t)} = -kc^2 \quad (9)}$$

evolution of the scale factor with time.

For a Universe with $k=0$ ($KF + PE = 0$) $\Rightarrow$ becomes
(Flat Universe)

$$H^2(t) - \frac{8}{3} \pi G \rho_c(t) = 0 \Rightarrow \boxed{\rho_c(t) = \frac{3}{8} \frac{H^2(t)}{\pi G} \quad (10)}$$

$\uparrow$ critical density

Let's calculate the critical density of the Universe at the present time: $\rho_c(t_0)$ c6

$$(10) \rightarrow \rho_c(t_0) = \frac{3}{8} \frac{H^2(t_0)}{\pi G}$$

$$H_0(t)_{\text{WMAP}} = 2.3 \times 10^{-18} \text{ s}^{-1}$$

$$\rightarrow \rho_c(t_0) = 9.47 \times 10^{-27} \text{ kg m}^{-3}$$

However WMAP measured the density due to baryonic matter (matter made up of protons, neutrons)

Baryons are made up of 3 quarks

Mesons are made up of 2 quarks

$$\text{to be } \rho_b(t_0) = 4.017 \times 10^{-28} \text{ kg m}^{-3} \approx \underline{\underline{0.04 \rho_c(t_0)}}$$

WMAP: Wilkinson Microwave Anisotropy Probe

Density Parameter

C7

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} \stackrel{(10)}{=} \frac{\frac{\rho(t)}{\frac{3}{8} \frac{H^2(t)}{\pi G}}}{\frac{3}{8} \frac{H^2(t)}{\pi G}} = \frac{8}{3} \frac{\rho(t) \pi G}{H^2(t)} \quad (11)$$

$$\Omega(t_0) = \frac{\rho(t_0)}{\rho_c(t_0)} = \frac{8}{3} \frac{\rho(t_0) \pi G}{H^2(t_0)}$$

↑
now

$$\Omega_{b,m}(t_0) = \frac{\rho_{b,m}(t_0)}{\rho_c(t_0)} = 0.044 \pm 0.004 \quad (\text{from WMAP})$$

↑

Baryonic
Density
Parameter
Now

Evolution of Density Parameter

$$\frac{\Omega(t)}{\Omega(t_0)} = \frac{\frac{\rho(t)}{H^2(t)} \cdot \frac{H^2(t_0)}{\rho(t_0)}}{\frac{\rho(t_0)}{H^2(t_0)}} = \frac{\rho(t)}{\rho(t_0)} \cdot \frac{H^2(t_0)}{H^2(t)} = (1+z)^3 \frac{H_0^2}{H^2(t)}$$

$$* \rho(t) = \rho_0 (1+z)^3$$

$$\boxed{\Omega(t) = \Omega_0 (1+z)^3 \frac{H_0^2}{H^2}} \quad (12)$$

Evolution of Density Parameter

c8

$$\textcircled{8} \Rightarrow H^2 R^2 - \frac{8\pi}{3} G \rho R^2 = -k c^2 \quad \left| \rightarrow H^2 R^2 - \frac{8\pi G R^2}{3} \left(\frac{3H^2 \Omega}{8\pi G} \right) = -k c^2 \right.$$

$$\textcircled{11} \Rightarrow \Omega(t) = \frac{8\pi G}{3} \frac{\rho}{H^2}$$

$$\Rightarrow H^2 R^2 - H^2 R^2 \Omega = -k c^2 \Rightarrow H^2 R^2 (1 - \Omega) = -k c^2 \quad \textcircled{13}$$

$$\text{for } t=t_0 \rightarrow H_0^2 (1 - \Omega_0) = -k c^2 \quad \textcircled{14}$$

$$\Rightarrow H^2 (1 - \Omega) = H_0^2 (1 - \Omega_0) (1+z)^2 \quad \textcircled{15}$$

$$(12) \wedge (15) \Rightarrow (1 - \Omega) = \frac{\Omega}{(1+z)^3 \Omega_0} \cdot (1 - \Omega_0) (1+z)^2$$

$$\Rightarrow (1+z) \Omega_0 (1 - \Omega) = \Omega (1 - \Omega_0)$$

$$\Rightarrow (\Omega_0 + z \Omega_0) (1 - \Omega) = \Omega - \Omega \Omega_0$$

$$\Rightarrow \Omega_0 + z \Omega_0 - \Omega_0 \Omega - z \Omega_0 \Omega = \Omega - \Omega \Omega_0$$

$$\Rightarrow \Omega \left[-\cancel{\Omega_0} - z \Omega_0 - 1 + \cancel{\Omega_0} \right] = -\Omega_0 - z \Omega_0$$

$$\Rightarrow \Omega = \frac{\Omega_0 (1+z)}{1 + z \Omega_0}$$

$$\Rightarrow \Omega = \frac{(1 + z \Omega_0) - (1 + z \Omega_0) + \Omega_0 (1+z)}{(1 + z \Omega_0)}$$

$$\Rightarrow \Omega = \frac{(1 + z \Omega_0) - (1 + z \Omega_0) + \Omega_0 (1+z)}{(1 + z \Omega_0)} \Rightarrow \boxed{\Omega = 1 + \frac{\Omega_0 - 1}{1 + z \Omega_0}} \quad \textcircled{16}$$

Example:

C9

When the Universe was 3 minutes old

(p^+) combined with (n) to form He

This happened at $z = 3.68 \times 10^8$

Assuming $\Omega_{0,m} = 0.27$

$$\Omega(z) = 1 + \frac{0.27 - 1}{(1 + 0.27 \cdot 3.68 \times 10^8)} \approx 1$$

for a Flat Universe $k=0$, $\Omega_0=1$, $\rho_0=\rho_{0,c}$

The equation of expansion of the Universe is.

$$(9) \rightarrow \left(\frac{dR}{dt} \right)^2 - \frac{8\pi G \rho_{0,c}}{3R} = -kc^2$$

$$\left(\frac{dR}{dt} \right)^2 = \frac{8\pi G \rho_{0,c}}{3R} \xrightarrow{\text{take } \sqrt{}} R^{1/2} \left(\frac{dR}{dt} \right) = \left(\frac{8\pi G \rho_{0,c}}{3} \right)^{1/2}$$

$$\Rightarrow \int R^{1/2} dR = \left(\frac{8\pi G \rho_{0,c}}{3} \right)^{1/2} \int dt$$

$$\frac{2}{3} R^{3/2} = \left(\frac{8\pi G \rho_{0,c}}{3} \right)^{1/2} t \Rightarrow \frac{2}{3} R^{3/2} = H_0 t$$

$$\Rightarrow R_{\text{flat}}^{3/2} = \frac{3}{2} H_0 t$$

$$\Rightarrow \boxed{R_{\text{flat}}(t) = \left(\frac{3}{2} \right)^{2/3} (H_0 t)^{2/3}} \quad (17)$$

$$t_H = \frac{1}{H_0} \quad R_{\text{flat}} = \left(\frac{3}{2} \right)^{2/3} \left(\frac{1}{t_H} \right)^{2/3}$$