

Age of a Dust filled Universe.

C 11

$$(17) \quad R_{\text{flat}}(t) = \left(\frac{3}{2}\right)^{2/3} \left(\frac{t}{t_H}\right)^{2/3} \quad \text{but } R(t) = \frac{1}{(1+z)}$$

$$\left(\frac{1}{1+z}\right) = \left(\frac{3}{2}\right)^{2/3} \left(\frac{t}{t_H}\right)^{2/3} \Rightarrow \left(\frac{t}{t_H}\right)^{2/3} = \frac{1}{(1+z)} \left(\frac{3}{2}\right)^{-2/3}$$

$$\Rightarrow \left(\frac{t}{t_H}\right) = \frac{1}{(1+z)^{3/2}} \left(\frac{3}{2}\right)^{-1} \quad (18)$$

Age of the Universe now ($z=0$)

$$\boxed{t_0 = \frac{2}{3} t_H} \quad (19)$$

Look Back Time: t_L

The look back time represents how far back in time we are looking when viewing an object at redshift z

$$t_L = t_0 - t(z) \quad (20)$$

$\uparrow$ age of Universe now $\nwarrow$ age of Universe at z

$$t_L = \frac{2}{3} t_H - \frac{t_H}{(1+z)^{3/2}} \cdot \frac{2}{3}$$

$$t_L = \frac{2}{3} t_H \left[1 - \frac{1}{(1+z)^{3/2}} \right] \quad (21)$$

A Universe that includes a fluid
with uniform, ρ, P, T

c12

We consider a Universe filled with a fluid of uniform density ρ , uniform pressure P and uniform temperature T . Since T is uniform there is no heat flow.

Let $r(t)$ be the radius of a comoving spherical surface. The 1st Law of Thermodynamics for this spherical volume containing the fluid is:

$$\underset{\substack{\uparrow \\ \text{change in} \\ \text{internal energy}}}{dU} = \underset{\substack{\uparrow \\ \text{heat} \\ \text{exchange}}}{\cancel{dQ}} - \underset{\substack{\uparrow \\ \text{work done} \\ \text{by fluid}}}{dW} \quad (22)$$

$$dW = dP \cdot V = P dV \quad \left| \quad V = \frac{4}{3} \pi r^3 \right. \rightarrow dW = \frac{4}{3} \pi P dr(t)^3$$

$$\text{Define } u = \frac{U}{V} \Rightarrow U = uV = \frac{4}{3} \pi r^3 u$$

$$(22) \Rightarrow \frac{4}{3} \pi \frac{d(r^3 u)}{dt} = - \frac{4}{3} \pi P \frac{dr^3}{dt}$$
$$\Rightarrow \frac{d(r^3 u)}{dt} = -P \frac{d(r^3)}{dt} \quad (23)$$

A Universe that includes a fluid of uniform, ρ, P, T

C13

$$\rho = \frac{m}{V} \quad \left| \begin{array}{l} E = mc^2 \end{array} \right. \rightarrow \rho = \frac{E}{c^2 V} = \frac{u}{c^2} \Rightarrow u = \rho c^2$$

$$(23) \Rightarrow c^2 \frac{d(\rho r^3)}{dt} = -P \frac{dr^3}{dt}$$

$$\Rightarrow c^2 \frac{d(\rho R^3)}{dt} = -P \frac{dR^3}{dt} \quad \text{FLUID EQUATION (24)}$$

For a pressureless Universe the fluid equation becomes:

$$\rho R^3 = \text{constant} = \rho(t_0) R(t_0)^3 = \rho_0$$

Derive the equation of acceleration of the Universe.

start from the expansion equation:

$$\left[\frac{1}{R^2} \left(\frac{dR}{dt} \right)^2 - \frac{8\pi G \rho}{3} \right] R^2 = -kc^2 \quad \text{Multiply both}$$

sides by R and differentiate:

$$R \left(\frac{dR}{dt} \right)^2 - \frac{8\pi G \rho R^3}{3} = -kc^2 R \quad (\text{take time derivative})$$

$$\left(\frac{dR}{dt} \right) \left(\frac{dR}{dt} \right)^2 + R \cdot 2 \left(\frac{dR}{dt} \right) \left(\frac{d^2 R}{dt^2} \right) - \frac{8\pi G}{3} \frac{d(\rho R^3)}{dt} = -kc^2 \frac{dR}{dt} \quad (\text{use fluid equation})$$

$$\frac{-8\pi G}{3} \frac{d(\rho R^3)}{dt} = + \frac{8\pi G}{3} \frac{P}{c^2} 3R^2 \frac{dR}{dt}$$

$$\cancel{\left(\frac{dR}{dt}\right)} \left(\frac{dR}{dt}\right)^2 + 2R \left(\frac{d^2R}{dt^2}\right) \cancel{\left(\frac{dR}{dt}\right)} + \frac{8\pi G}{3} \frac{P}{c^2} \cancel{3R^2} \cancel{\left(\frac{dR}{dt}\right)} = -kc^2 \cancel{\left(\frac{dR}{dt}\right)} \quad (\text{Use expansion equation again})$$

$$\cancel{\left(\frac{dR}{dt}\right)^2} + 2R \left(\frac{d^2R}{dt^2}\right) + \frac{8\pi G}{c^2} P R^2 = \cancel{\left(\frac{dR}{dt}\right)^2} - \frac{8}{3} \pi G \rho R^2$$

$$\frac{d^2R}{dt^2} = - \frac{4\pi G}{c^2} \rho R - \frac{4}{3} \pi G P R$$

$$\frac{d^2R}{dt^2} = - \frac{4\pi G R}{3} \left[\rho + \frac{3P}{c^2} \right] \quad \begin{matrix} 25 \\ \text{Acceleration} \\ \text{Equation.} \end{matrix}$$

Notice that a positive pressure contributes to a negative acceleration (i.e. slows down the expansion!). There are no pressure gradients so the pressure does not "push" outwards. Its only effect is to increase the energy density of the Universe making gravity stronger.

SUMMARY : Universe

C15

with fluid of constant ρ, P, T

Acceleration Equation

$$\frac{d^2 R}{dt^2} = -\frac{4\pi G}{3} \left(\rho + \frac{3P}{c^2} \right) R \quad (26)$$

Fluid Equation

$$\frac{d(R^3 \rho)}{dt} = -\frac{P}{c^2} \frac{dR^3}{dt} \quad (27)$$

Expansion Equation

$$\left[\frac{1}{R^2} \left(\frac{dR}{dt} \right)^2 - \frac{8\pi G \rho}{3} \right] R^2 = -kc^2 \quad (28)$$

Equation of state: $P = wu = w\rho c^2$ (29)

$$u = \frac{U}{V} = \frac{mc^2}{V} = \rho c^2$$

(29) $\wedge$ (27)

$$3R^2 \frac{d\rho}{dt} \rho + R^3 \frac{d\rho}{dt} = - \frac{w\rho c^2}{c^2} 3R^2 \frac{d\rho}{dt} \Rightarrow$$

$$3R^2 \frac{dR}{dt} \rho (1+w) = - R^3 \frac{d\rho}{dt} \rightarrow$$

$$3 \frac{1}{R} \frac{dR}{dt} (1+w) = - \frac{1}{\rho} \frac{d\rho}{dt} \rightarrow$$

$$\frac{d \ln R}{dt} = \frac{d \ln R}{dR} \frac{dR}{dt} = \frac{1}{R} \frac{dR}{dt} \quad \text{and} \quad \frac{d \ln \rho}{dt} = \frac{1}{\rho} \frac{d\rho}{dt}$$

$$3 \frac{d \ln R}{dt} (1+w) = - \frac{d \ln \rho}{dt} \Rightarrow$$

$$3 \int d \ln R (1+w) = - \int d \ln \rho \Rightarrow$$

$$3 (\ln R) (1+w) = - \ln \rho + \text{constant} \Rightarrow$$

$$\ln R^{3(1+w)} + \ln \rho = \text{constant} \Rightarrow$$

$$\ln \rho \cdot R^{3(1+w)} = \text{constant}$$

FLUID

EQUATION

$$\rightarrow \left[\rho R^{3(1+w)} = \text{constant} \right] = \rho_0 \quad (30)$$

Deceleration Parameter

$$q(t) = - \frac{R(t) \frac{d^2 R}{dt^2}}{\left[\frac{dR}{dt} \right]^2}$$

(31)

Cooling of the Universe after the Big Bang C18

Start from fluid equation: $R^{3(1+w)} \rho_{\text{rad}} = \rho_{\text{rad}}(t_0)$

$$u = \frac{U}{V} = \frac{nc^2}{V} = \rho c^2 \Rightarrow \rho = \frac{u}{c^2}$$

$$R^{3(1+w)} \frac{u_{\text{rad}}(t)}{c^2} = \frac{u_{\text{rad}}(t=0)}{c^2} \Rightarrow$$

For Blackbody Radiation: $u_{\text{rad}} = aT^4$, where $a = \frac{4\sigma}{c}$
and σ is the
S-B constant

$$R^{3(1+w)} aT^4 = aT_0^4$$

For Blackbody Radiation: $w = \frac{1}{3}$

$$R^4 aT^4 = aT_0^4 \Rightarrow RT = T_0 \Rightarrow \boxed{T = (1+z)T_0}$$

(32)

An Estimate of the Present Blackbody CMB temperature of the Universe

<19

The temperature and density required to fuse $H \Rightarrow He$ in the early Universe are:

$$T \simeq 10^9 K \text{ and } \rho_b \simeq 10^{-2} \text{ kg m}^{-3}$$

↑
baryonic density.

From WMAP we know $\rho_{b,0} = 4.17 \times 10^{-28} \text{ kg m}^{-3}$
↑
baryonic density now

The fluid equation for baryonic density evolution ($w=0$)

$$\rho_b R(t) = \rho_{b,0} \Rightarrow R = \left(\frac{\rho_{b,0}}{\rho_b} \right)^{1/3} = \left(\frac{4.17 \times 10^{-28} \text{ kg m}^{-3}}{10^{-2}} \right)^{1/3} = 3.47 \times 10^{-9}$$

From the temperature evolution equation:

$$RT = T_0 \Rightarrow T_0 = RT = (3.47 \times 10^{-9}) \cdot 10^9 K = 3.47 K$$

The peak wavelength that corresponds to this temperature for blackbody radiation is:

$$\lambda_p = \frac{0.0029 \text{ Km}}{3.47 K} = 8.36 \times 10^{-4} \text{ m}$$

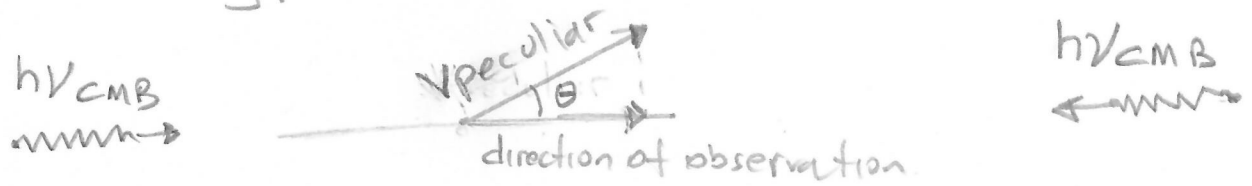
c20

Penzias and Wilson discovered the blackbody radiation that fills the Universe. This blackbody spectrum peaks at $\lambda_p = 1.06 \text{ mm}$. This afterglow of the Big Bang is known as the COSMIC MICROWAVE BACKGROUND (CMB).

The CMB does not emanate from any specific object but it is present in the entire Universe.

All observers see the same spectrum (for observers moving with the Hubble Flow).

For observers moving with respect to the Hubble flow (the velocity with respect to the Hubble Flow is called peculiar velocity) there is a Doppler shift:



$$\text{For } v \ll c \quad \frac{\lambda_{\text{obs}} - \lambda_0}{\lambda_0} = \frac{-v_p \cos \theta}{c} \Rightarrow \frac{\lambda_{\text{obs}}}{\lambda_0} - 1 = -\frac{v_p \cos \theta}{c}$$

$$\lambda_{\text{obs}} = \lambda_0 \left(1 - \frac{v_p \cos \theta}{c} \right) \Rightarrow \frac{1}{T_{\text{obs}}} = \frac{1}{T_{\text{rest}}} \left(1 - \frac{v_p \cos \theta}{c} \right)$$

$$T_{\text{obs}} = \frac{T_{\text{rest}}}{\left(1 - \frac{v_p \cos \theta}{c} \right)} \approx T_{\text{rest}} \left(1 + \frac{v_p \cos \theta}{c} \right) \quad (33)$$

$$\text{For } v \sim c \quad T_{\text{obs}} = \frac{T_{\text{rest}} \sqrt{1 - \frac{v^2}{c^2}}}{\left(1 - \frac{v_p \cos \theta}{c} \right)} \quad (34)$$

The sun's motion through the CMB C21

The SUN is moving with a speed (peculiar velocity) of $V_p \approx 370 \text{ km/s}$ towards $(\alpha, \delta) = (11.2^\text{h}, -7^\circ)$.

Once the anisotropy caused by our peculiar velocity with respect to the Hubble Flow is removed, the CMB looks very uniform in all directions.

Sensitive measurements (WMAP) have measured small deviations of the temperature: T_{CMB} of ~ 1 part in 10^5 .

Analysis of the small scale fluctuations of the CMB have led to very precise constraints on the cosmological parameters.

The Sunyaev-Zel'dovich Effect

When a CMB photon interacts with an e^- in the hot intracluster gas ($\approx 10^8 \text{ K}$) of rich clusters of galaxies it is scattered to a higher energy (Inverse Compton Scattering)

The increase in frequency is:

$$\text{change in frequency} \rightarrow \frac{\Delta \nu}{\nu_{\text{original}}} = \frac{4 k T_e}{m_e c^2} \leftarrow \begin{array}{l} \text{electron temperature} \\ \text{of hot gas} \end{array}$$

Calculation of Interaction time of a photon with an electron prior to recombination.

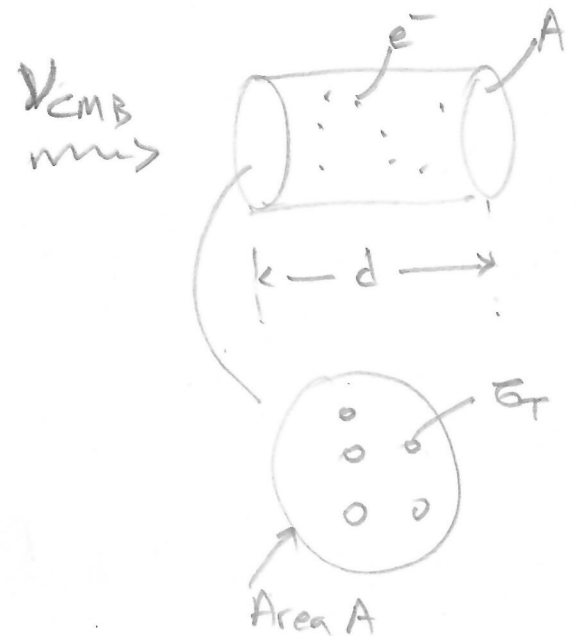
C 22

The probability for an interaction is

$$P = \frac{N \cdot \sigma_T}{A} = \frac{n V \sigma_T}{A}$$

$$\rho_p = \frac{N m_p}{V} = n_p m_p \rightarrow n_p = \frac{\rho_p}{m_p}$$

$$n_p = n_e$$



$$P = \frac{\rho_p V \sigma_T}{A m_p} = \frac{\rho_p (d \cdot A) \sigma_T}{A m_p} = \frac{\rho_p d \sigma_T}{m_p} \quad (36)$$

The distance (L) travelled between collisions is called the mean free path (set $P=1$)

$$L = \frac{m_p}{\rho_p \sigma_T} \quad (37)$$

The time between collisions:

$$t_{\text{scat}} \sim \frac{L}{c} = \frac{m_p}{\rho_p \sigma_T \cdot c} = \frac{m_p R^3}{\rho_0 \sigma_T \cdot c}$$

(38)

(collisions became very rare as the Universe expands)

The timescale of the expansion of the Universe C23

$$* V(t) = H(t) r(t) \rightarrow \frac{dR(t)}{dt} = H(t) R(t)$$

$$\left(\frac{1}{R} \frac{dR}{dt} \right) = H(t)$$

$$t_{\text{expansion}} \approx \left[\frac{1}{R} \frac{dR}{dt} \right]^{-1} = \frac{1}{H(t)} \quad (39)$$

Decoupling of light from matter occurs when $t_{\text{scat}} \gtrsim t_{\text{expansion}}$ or :

$$\frac{m_p R^3}{\rho_0 \cdot \sigma_T \cdot c} \gtrsim \frac{1}{H(t)} \rightarrow R^3 \gtrsim \frac{\rho_0 \cdot \sigma_T \cdot c}{m_p H(t)} \quad (40)$$

$(t \approx 20 \times 10^6 \text{ years})$

But photon decoupling happened much earlier when e^- combined with p^+ to form neutral H atoms. This is referred to as recombination.

Photons of the CMB come from the era of recombination. We can't see further back.

$z_{\text{recombination}} \approx 1100$ (380,000 years after Big Bang)

We can incorporate the contribution of relativistic particles (CMB photons and neutrinos) to the expansion of the Universe by considering the equivalence between energy and mass.

We now add CMB photons and neutrinos to our model Universe.

ρ_m : density of matter (baryonic and dark)

ρ_{rel} : density of relativistic particles (photons & neutrinos)

(41) Equation of State $P = w \cdot \rho$

$\uparrow$
pressure

$\uparrow$
energy density

$w_m = 0$

 $w_{rel} = \frac{1}{3}$

 $w_\Lambda = -1$
 $\uparrow$
dark energy

The FLUID EQUATION $\frac{d\rho}{dt} + \frac{3(1+w)}{R} \rho = 0$

Indicates that the density of matter, relativistic particles, and dark energy will evolve differently.

The expansion equation for matter and relativistic particles:

$$\left[\left(\frac{1}{R} \frac{dR}{dt} \right)^2 - \frac{8\pi G}{3} (\rho_m + \rho_{rel}) \right] R^2 = -kc^2 \quad (42)$$

$$\Omega_m = \frac{\rho_m}{\rho_c} = \frac{\rho_m}{\left(\frac{3H^2}{8\pi G} \right)} = \frac{8\pi G \rho_m}{3H^2}$$

$$\Omega_{rel} = \frac{\rho_{rel}}{\rho_c} = \frac{8\pi G \rho_{rel}}{3H^2}$$

$$V(t) = H(t) r(t) \rightarrow \frac{dR}{dt} = H(t) R(t) \rightarrow \frac{1}{R} \frac{dR}{dt} = H(t)$$

$$(42) \rightarrow H(t)^2 R^2 - \frac{8\pi G}{3} \left[\frac{3H^2}{8\pi G} \Omega_m + \frac{3H^2}{8\pi G} \Omega_{rel} \right] R^2 = -kc^2$$

$$H^2 \left[1 - (\Omega_m + \Omega_{rel}) \right] R^2 = -kc^2$$

For a Flat Universe $k=0 \rightarrow \Omega_m + \Omega_{rel} = 1$

Density Parameter of relativistic particles now $\Omega_{rel,0}$

$$\Omega_{rel,0} = \frac{8\pi G \rho_{rel,0}}{3H_0^2} = \frac{8\pi G}{3H_0^2} \frac{u_{rel,0}}{c^2} = \frac{8\pi G}{3H_0^2} \frac{(1.68 a T_0^4)}{c^2} \quad (43)$$

$$\Omega_{rel,0} = 8.24 \times 10^{-5}, \text{ for } T_0 = 2.725 K$$

$$\Omega_{m,0} = 0.27$$

Transition from Radiation Dominated to Matter dominated Universe

we use the fluid equation $R^{3(w+1)} \rho = \rho_0$

$$w=0 \text{ for matter} \rightarrow R^3 \rho_m = \rho_{m,0}$$

$$w=\frac{1}{3} \text{ for radiation} \rightarrow R^4 \rho_{rel} = \rho_{rel,0}$$

The transition from radiation dominated to matter dominated Universe happens when: $\rho_m = \rho_{rel}$

$$\frac{\rho_{rel}}{\rho_m} = 1 \rightarrow \frac{\rho_{rel,0}}{R^4} \cdot \frac{R^3}{\rho_{m,0}} = 1 \rightarrow$$

$$\rightarrow R_{r,m} = \frac{\rho_{rel,0}}{\rho_{m,0}} = \frac{\Omega_{rel,0}}{\Omega_{m,0}} = \frac{8.24 \times 10^{-5}}{0.27}$$

$$\rightarrow R_{r,m} = 3.05 \times 10^{-4} \rightarrow \frac{1}{1 - z_{r,m}} = 3.05 \times 10^{-4}$$

$$|z_{r,m} = 3270| \quad (44)$$

$$T(z) = (1+z)T_0 = (1+3270) \times 2.725 \text{ K} = \underline{\underline{8920 \text{ K}}}$$

Age of the 2 Component Universe

c27

The expansion equation for a 2 component Universe is:

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G}{3} (\rho_m + \rho_{rel}) R^2 = -kc^2$$

where $\rho_m = \frac{\rho_{m,0}}{R^3}$ and $\rho_{rel} = \frac{\rho_{rel,0}}{R^4}$

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G}{3} \left(\frac{\rho_{m,0}}{R^3} + \frac{\rho_{rel,0}}{R^4} \right) R^2 = -kc^2$$

To find the age of the Universe at some scale factor R we multiply the above equation by R^2 , or take the square root and then integrate both sides. (for flat Universe $k=0$)

$$R \left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3} (R \rho_{m,0} + \rho_{rel,0})$$

$$\int \frac{R dR}{\sqrt{R \rho_{m,0} + \rho_{rel,0}}} = \sqrt{\frac{8\pi G}{3}} \int dt \rightarrow \boxed{\text{EQ 29.81 from CRO}}$$

Solving Einstein's Equations for an isotropic, homogeneous Universe leads to the Friedmann Equation:

$$\left[\left(\frac{1}{R} \frac{dR}{dt} \right)^2 - \frac{8\pi G}{3} \rho \right] R^2 = -kc^2 \quad (45)$$

Einstein realized that as originally conceived his field equations did not describe a static Universe so he introduced an ad hoc term called the cosmological constant Λ .

$$\left[\left(\frac{1}{R} \frac{dR}{dt} \right)^2 - \frac{8\pi G}{3} \rho - \frac{1}{3} \Lambda c^2 \right] R^2 = -kc^2 \quad (46)$$

The Universe was found by observations to be expanding and not static, however, the cosmological constant Λ is now used in the Friedmann equation to describe the acceleration of the expansion as produced by Dark Energy.

The acceleration of the Universe was indicated by observations made in the late 1990's of distant SN type Ia's.

FRIEDMANN EQUATION

$$(46) \Rightarrow \left[\left(\frac{1}{R} \frac{dR}{dt} \right)^2 - \frac{8\pi G}{3} (\rho_m + \rho_{rel} + \rho_\Lambda) \right] R^2 = -kc^2 \quad (47)$$

$$\text{where we set } \frac{8\pi G}{3} \rho_\Lambda = \frac{\Lambda c^2}{3} \Rightarrow \rho_\Lambda = \frac{\Lambda c^2}{8\pi G} \quad (48)$$

We derive the acceleration equation by multiplying equation (47) by R and integrating with respect to t .

Acceleration Equation

$$\frac{d^2 R}{dt^2} = - \frac{4\pi G R}{3} \left[\rho_m + \rho_{rel} + \rho_\Lambda + \frac{3(\rho_m + \rho_{rel} + \rho_\Lambda)}{c^2} \right]$$

$$\text{where } \rho_\Lambda = \frac{\Lambda c^2}{8\pi G}, \quad P_\Lambda = -\rho_\Lambda c^2$$

Equation of state

$$P = w \rho c^2$$

$$P_m = 0, \quad w_m = 0$$

$$P_{rel} = \frac{1}{3} \rho_{rel} c^2, \quad w_{rel} = \frac{1}{3}$$

$$P_\Lambda = -\rho_\Lambda c^2, \quad w_\Lambda = -1$$

Critical Density $\rho_c = \frac{3H^2}{8\pi G}$, $H = \frac{1}{R} \frac{dR}{dt}$

$$\Omega_m = \frac{\rho_m}{\rho_c} = \rho_m \frac{8\pi G}{3H^2}$$

$$\Omega_{rel} = \frac{\rho_{rel}}{\rho_c} = \rho_{rel} \frac{8\pi G}{3H^2}$$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \rho_\Lambda \frac{8\pi G}{3H^2}$$

Friedmann's Equation becomes

$$\left[H^2 - H^2 (\Omega_m + \Omega_{rel} + \Omega_\Lambda) \right] R^2 = -kc^2$$

for a flat Universe $k=0 \Rightarrow$

$$\boxed{\Omega_m + \Omega_{rel} + \Omega_\Lambda = 1}$$

$$\text{WMAP} = \begin{cases} \Omega_{m,0} = 0.27 \pm 0.04 \\ \Omega_{rel,0} = 8.24 \times 10^{-5} \\ \Omega_{\Lambda,0} = 0.73 \pm 0.04 \\ \Omega_0 = 1.02 \pm 0.02 \end{cases}$$

The Universe is flat!

FLUID EQUATION

C31

$$R^{3(w+1)} \rho = \rho_0$$

$$\rho_m = \rho_{m,0} R^{-3} = \rho_{m,0} (1+z)^3$$

$$\rho_{rel} = \rho_{rel,0} R^{-4} = \rho_{rel,0} (1+z)^4$$

$$\rho_\Lambda = \rho_{\Lambda,0} = \text{constant.}$$

The early Universe was radiation dominated (radiation era)

Present Universe is Dark Energy dominated (Λ era)

In between the Universe was matter dominated (matter era)

When did the matter density equal the dark energy density? $\rho_m = \rho_\Lambda$

$$\rho_{m,0} (1+z)^3 = \rho_{\Lambda,0} \Rightarrow (1+z) = \left(\frac{\rho_{\Lambda,0}}{\rho_{m,0}} \right)^{1/3}$$

$$(1+z) = \left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} \right)^{1/3} = \left(\frac{0.73}{0.27} \right)^{1/3} \Rightarrow \boxed{z_{m,\Lambda} \approx 0.39}$$